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A Fiber Integration Formula for the Smooth Deligne Cohomology

Kiyonori Gomi and Yuji Terashima

1 Introduction

As is well known, an important operation exists in de Rham cohomology called *integration along the fiber* for a fiber bundle. In this paper, we extend it to the *smooth Deligne cohomology*, which is a refinement of ordinary cohomology. In the interesting work [BM], Brylinski and McLaughlin show in an abstract way that there exists such an operation for a fiber bundle with a Riemann surface as a fiber, and they use it to construct the Quillen metric on the determinant line bundle over the moduli space of stable bundles on a Riemann surface, which has been recently extended to the general case in [B2] (see also [GS]). In this paper we construct such an *explicit* map on the Čech *cochains* and prove a Stokes-type formula which shows that it is a chain map when the fiber has no boundary. Moreover, we prove that the induced map on the smooth Deligne cohomology is *canonical*; that is, it is independent of all choices used in the construction. This construction is performed along ideas by Brylinski [B1] and Gawedzki [G]. It seems to be an interesting but difficult problem to interpret our results in terms of the differential geometry of higher gerbes, which are geometric objects corresponding to elements of the smooth Deligne cohomology (see [B1], [B2], and [CMW]). We have just heard that an integration theory of the Deligne cocycles was developed and then applied to the Chern-Simons theory, independently from and before us, by Freed [F].

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2 Definition of the integration map

In this section we construct the integration map along the fiber for the smooth Deligne cohomology. For simplicity, we consider only product bundles in the sequel.

Let X be a smooth manifold. The smooth Deligne cohomology of X is defined as the hypercohomology of the complex of sheaves

$$\underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_X \xrightarrow{d} \cdots \xrightarrow{d} \underline{A}^p_X,$$

where $\underline{A}^k_X$ denotes the sheaf of smooth complex-valued k-forms on X (see [B1]).

The hypercohomology of a complex of sheaves is a useful generalization of the cohomology of a single sheaf. As with usual sheaf cohomology, there is a Čech definition, which is used in the sequel. For any open covering $\mathfrak{U}$ of X and for any complex of sheaves $\mathcal{F}^\bullet$, the cochains $C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)$ form a double complex, one differential δ coming from the covering and the other from the complex of sheaves $\mathcal{F}^\bullet$. The Čech hypercohomology groups are defined by

$$H^q\left(X, \mathcal{F}^\bullet\right) = \varinjlim_{\mathfrak{U}} H^q\left(C^\bullet\left(\mathfrak{U}, \mathcal{F}^\bullet\right)\right),$$

where $H^q(C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet))$ denotes the cohomology of the single complex with differential, denoted by D, associated in the usual way to the double complex.

Now we consider a product bundle $\pi : X \times M \to X$, where X is a paracompact smooth manifold, and M is a compact oriented smooth manifold of dimension m. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$, and let $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$ be locally finite good coverings of X and M, respectively. An open cover $\mathfrak{U} = \{U_\alpha\}$ is called good cover if all nonempty intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are contractible. Hence, $\mathfrak{W} = \{U_\alpha \times V_\beta\}_{(\alpha,\beta) \in I \times J}$ is a good covering of $X \times M$.

To construct the integration map along the fiber M, we first choose a triangulation $t : |K| \to M$ of M; that is, $|K|$ is a polyhedron associated with a simplicial complex K and t is a homeomorphism. We assume that for any simplex $\sigma \in K$, there is an open set $V \in \mathfrak{V}$ such that $t(|\sigma|) \subset V$; that is, a triangulation (t, K) is *finer* than the covering $\mathfrak{V}$. We then fix a map $\phi : K \to J$ such that $t(|\sigma|) \subset V_{\phi(\sigma)}$ (called an *index map*).

We define the set of flags of simplices $F(i)$ by

$$F(i) = \left\{\vec{\sigma} = \left(\sigma^{m-i}, \ldots, \sigma^m\right) \,\middle|\, \sigma^p \in K,\ \dim \sigma^p = p,\ \sigma^{m-i} \subset \cdots \subset \sigma^m\right\}$$

and the set of sequences of integers $P_q(i)$ by

$$P_q(i) = \left\{\vec{n} = \left(n_1, \ldots, n_q\right) \,\middle|\, n_j \in \mathbb{Z},\ m \geq n_1 \geq \cdots \geq n_q \geq m - i\right\}.$$

In the sequel, it is convenient to put $n_0 = m$, $n_{q+1} = m - i$, and $|\vec{n}| = \sum_{s=1}^q n_s$.

Definition 2.1. The integration map along the fiber M,

$$\psi_{K,\phi} : C^k\left(\mathfrak{W}, \underline{\mathbb{C}}^*_{X\times M} \xrightarrow{d\log} \underline{A}^1_{X\times M} \longrightarrow \cdots \longrightarrow \underline{A}^p_{X\times M}\right) \longrightarrow C^{k-m}\left(\mathfrak{U}, \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_X \longrightarrow \cdots \longrightarrow \underline{A}^{p-m}_X\right),$$

is defined by

$$\psi_{K,\phi}\left(f, \omega^1, \ldots, \omega^p\right) = \left(g, \theta^1, \ldots, \theta^{p-m}\right),$$

where

$$g_{\alpha_0\cdots\alpha_{k-m}} = \exp\left(\sum_{i=0}^{m-1}\sum_{\vec{\sigma}\in F(i)}\sum_{\vec{n}\in P_{k-m}(i)}(-1)^{|\vec{n}|}\int_{\sigma^{m-i}}\omega^{m-i}_{\gamma^0_{\sigma^m}\cdots\gamma^0_{\sigma^{n_1}}\gamma^1_{\sigma^{n_1}}\cdots\gamma^1_{\sigma^{n_2}}\cdots\gamma^{k-m}_{\sigma^{n_{k-m}}}\cdots\gamma^{k-m}_{\sigma^{m-i}}}\right) \times \prod_{\vec{\sigma}\in F(m)}\prod_{\vec{n}\in P_{k-m}(m)}\left(\int_{\sigma^0} f_{\gamma^0_{\sigma^m}\cdots\gamma^0_{\sigma^{n_1}}\gamma^1_{\sigma^{n_1}}\cdots\gamma^1_{\sigma^{n_2}}\gamma^2_{\sigma^{n_2}}\cdots\gamma^2_{\sigma^{n_3}}\cdots\gamma^{k-m}_{\sigma^{n_{k-m}}}\cdots\gamma^{k-m}_{\sigma^0}}\right)^{(-1)^{|\vec{n}|}}$$

and

$$\theta^l_{\alpha_0\cdots\alpha_{k-m-l}} = \sum_{i=0}^{m}\sum_{\vec{\sigma}\in F(i)}\sum_{\vec{n}\in P_{k-m-l}(i)}(-1)^{|\vec{n}|}\int_{\sigma^{m-i}}\omega^{m+l-i}_{\gamma^0_{\sigma^m}\cdots\gamma^0_{\sigma^{n_1}}\gamma^1_{\sigma^{n_1}}\cdots\gamma^1_{\sigma^{n_2}}\cdots\gamma^{k-m-l}_{\sigma^{n_{k-m-l}}}\cdots\gamma^{k-m-l}_{\sigma^{m-i}}}.$$

Here, the index $\gamma^r_{\sigma^q} = (\alpha_r, \phi(\sigma^q)) \in I \times J$.

We note that an orientation of M (hence, that of top-dimensional simplex σ^m) induces that of σ^q for $q = m-i, \ldots, m$ along the flag $\sigma^{m-i} \subset \cdots \subset \sigma^m$. In particular, the induced orientation of σ^{m-i} gives the integration $\int_{\sigma^{m-i}}$.

3 Main theorems

In this section we prove the Stokes-type formula for the fiber integration $\psi_{K,\phi}$, which shows that $\psi_{K,\phi}$ is a chain map when the fiber has no boundary. We prove that the induced map in the cohomology is independent of all choices used in the construction of $\psi_{K,\phi}$.

Theorem 3.1. The fiber integration $\psi_{K,\phi}$ satisfies

$$\psi_{K,\phi} \circ D = (-1)^m D \circ \psi_{K,\phi} + (\partial\psi)_{\partial K,\partial\phi},$$

where $(\partial\psi)_{\partial K,\partial\phi}$ is the fiber integration for the product bundle $X \times \partial M \to X$, with the restrictions to the boundary ∂M of the triangulation and the index map. □

Corollary 3.2. If the fiber M has no boundary, then the map $\psi_{K,\phi}$ is a chain map. □

Proof. We put

$$\left(h, \tau^1, \dots, \tau^{p-m}\right) = \psi_{K,\phi} \circ D\left(f, \omega^1, \dots, \omega^p\right)$$

and

$$\left(h', {\tau'}^1, \dots, {\tau'}^{p-m}\right) = D \circ \psi_{K,\phi}\left(f, \omega^1, \dots, \omega^p\right).$$

By the definition of the integration map $\psi_{K,\phi}$, we have

$$\begin{aligned}
&\tau^{l+1}_{\alpha_0 \cdots \alpha_{k-m-l}} - (-1)^m {\tau'}^{l+1}_{\alpha_0 \cdots \alpha_{k-m-l}} \\
&\quad = \sum_{i=0}^{m} \sum_{\vec{\sigma} \in F(i)} \sum_{\vec{n} \in P_{k-m-l}(i)} (-1)^{|\vec{n}|} \int_{\sigma^{m-i}} \left(\delta\omega^{m+l+1-i}\right)_{\gamma^0_{\sigma^m} \cdots \gamma^0_{\sigma^{n_1}} \gamma^1_{\sigma^{n_1}} \cdots \gamma^1_{\sigma^{n_2}} \cdots \gamma^{k-m-l}_{\sigma^{m-i}}} \\
&\qquad + \sum_{i} \sum_{\vec{\sigma}} \sum_{\vec{n}} (-1)^{|\vec{n}|+k-m-l-i} \\
&\qquad\qquad \times \left(\int_{\sigma^{m-i}} d\omega^{m+l-i}_{\gamma^0_{\sigma^m} \cdots \gamma^{k-m-l}_{\sigma^{m-i}}} - (-1)^{m-i} d \int_{\sigma^{m-i}} \omega^{m+l-i}_{\gamma^0_{\sigma^m} \cdots \gamma^{k-m-l}_{\sigma^{m-i}}} \right) \\
&\qquad + \left(\delta\theta^{l+1}\right)_{\alpha_0 \cdots \alpha_{k-m-l}}.
\end{aligned} \tag{3.1}$$

We split the terms on the first line of the right-hand side of equation (3.1)

$$\begin{aligned}
&\sum_{i=0}^{m} \sum_{\vec{\sigma} \in F(i)} \sum_{\vec{n} \in P_{k-m-l}(i)} (-1)^{|\vec{n}|} \\
&\qquad \times \int_{\sigma^{m-i}} \left(\delta\omega^{m+l+1-i}\right)_{\gamma^0_{\sigma^m} \cdots \gamma^{j-1}_{\sigma^{n_j}} \gamma^j_{\sigma^{n_j}} \cdots \gamma^j_{\sigma^{n_{j+1}}} \gamma^{j+1}_{\sigma^{n_{j+1}}} \cdots \gamma^{k-m-l}_{\sigma^{m-i}}} \\
&\quad = \sum_{i=0}^{m} \sum_{\vec{\sigma} \in F(i)} \sum_{\vec{n} \in P_{k-m-l}(i)} (-1)^{|\vec{n}|} \\
&\qquad \times \int_{\sigma^{m-i}} \sum_{j=0}^{k-m-l} \sum_{q=0}^{n_{j+1}-n_j} (-1)^{j+q} \omega^{m+l+1-i}_{\cdots \gamma^j_{\sigma^{n_j}} \cdots \widehat{\gamma^j_{\sigma^{n_j-q}}} \cdots \gamma^j_{\sigma^{n_{j+1}}} \cdots}
\end{aligned}$$

into three pieces:

(i) $j = r$ and $q = 0$ for r with $n_{r+1} - n_r = 0$ so that the γ_*^r vanish;

(ii) $j = r$ and $q = 0$ or $q = n_{r+1} - n_r$ for r with $n_{r+1} - n_r > 0$;

(iii) all other terms.

The first piece cancels with the third line of the right-hand side of equation (3.1). In the second piece, all terms, with the exception of the terms with $j = 0$, $q = 0$ and the terms with $j = k - m - l$, $q = m - n_{k-m-l}$, vanish because terms with $j = r - 1$, $q = n_r$ and terms with $j = r$, $q = n_{r-1}$ cancel in pairs. The terms with $j = 0$, $q = 0$ are equal to the element η^{l+1}, where we put

$$\left(s, \eta^1, \dots, \eta^{p-m}\right) = (\partial\psi)_{\partial K, \partial\varphi}\left(f, \omega^1, \dots, \omega^p\right)$$

since for each $(m-1)$-dimensional simplex $\sigma^{m-1} \in K \setminus \partial K$, there exist just two top-dimensional simplices which have σ^{m-1} as a face and induce opposite orientations. The terms with $j = k - m - l$, $q = m - n_{k-m-l}$ cancel those on the second line of the right side of equation (3.1) by the formula for a differential form ω,

$$\int_{\sigma^{m-i}} d\omega = (-1)^{m-i}\, d\int_{\sigma^{m-i}} \omega + \int_{\partial\sigma^{m-i}} \omega.$$

The third piece vanishes because for each partial flag

$$\sigma^{m-i} \subset \cdots \subset \sigma^{s-1} \subset \widehat{\sigma^s} \subset \sigma^{s+1} \subset \cdots \subset \sigma^m,$$

there exist just two flags

$$\sigma^{m-i} \subset \cdots \subset \sigma^{s-1} \subset \sigma_+^s \subset \sigma^{s+1} \subset \cdots \subset \sigma^m$$

and

$$\sigma^{m-i} \subset \cdots \subset \sigma^{s-1} \subset \sigma_-^s \subset \sigma^{s+1} \subset \cdots \subset \sigma^m$$

which extend it and induce opposite orientations of σ^{m-i}.

Hence, we have $\tau^{l+1} = (-1)^m {\tau'}^{l+1} + \eta^{l+1}$. Similarly, we can show $f = {f'}^{(-1)^m} \times s$, which completes the proof of Theorem 3.4. ■

Remark 3.1. The sign $(-1)^m$ in the above theorem corresponds to that in the usual formula $\int_M d\omega = (-1)^m\, d\int_M \omega + \int_{\partial M} \omega$ for the integration along the fiber M of a differential form ω on $X \times M$.

In the sequel, we suppose that the fiber M has no boundary. Then by Corollary 3.2 and the fact that $\mathfrak{U}$ and $\mathfrak{W}$ are both good coverings, we find that the map $\psi_{K,\phi}$ induces the map on the smooth Deligne cohomology

$$\psi^{\#}_{K,\phi} : H^k\Big(X \times M, \underline{\mathbb{C}}^*_{X\times M} \xrightarrow{d\log} \underline{A}^1_{X\times M} \longrightarrow \cdots \longrightarrow \underline{A}^p_{X\times M}\Big) \longrightarrow H^{k-m} \Big(X, \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_X \longrightarrow \cdots \longrightarrow \underline{A}^{p-m}_X\Big).$$

To prove that the map $\psi^{\#}_{K,\phi}$ is independent of the choice of index map, for two index maps $\phi, \phi' : K \to J$, we define the map

$$H : C^k\Big(\mathfrak{W}, \underline{\mathbb{C}}^*_{X\times M} \xrightarrow{d\log} \underline{A}^1_{X\times M} \longrightarrow \cdots \longrightarrow \underline{A}^p_{X\times M}\Big) \longrightarrow C^{k-m-1} \Big(\mathfrak{U}, \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_X \longrightarrow \cdots \longrightarrow \underline{A}^{p-m}_X\Big)$$

by

$$H\big(f, \omega^1, \ldots, \omega^p\big) = \big(h, \eta^1, \ldots, \eta^{p-m}\big),$$

where

$$h_{\alpha_0\cdots\alpha_{k-m-1}} = \exp\Bigg(\sum_{i=0}^{m-1} \sum_{\vec{\sigma}\in F(i)} \sum_{\vec{n}\in P_{k-m-1}(i)} \sum_{j=0}^{k-m-1} \sum_{q=0}^{n_{j+1}-n_j} (-1)^{|\vec{n}|+j+q} \times \int_{\sigma^{m-i}} \omega^{m-i}_{\gamma^0_{\sigma^m}\cdots\gamma^j_{\sigma^{n_j-q}}\gamma'^j_{\sigma^{n_j-q}}\cdots\gamma'^{k-m-1}_{\sigma^{m-i}}}\Bigg) \times \prod_{\vec{\sigma}\in F(m)} \prod_{\vec{n}\in P_{k-m-1}(m)} \prod_{j=0}^{k-m-1} \prod_{q=0}^{n_{j+1}-n_j} \bigg(\int_{\sigma^0} f_{\gamma^0_{\sigma^m}\cdots\gamma^j_{\sigma^{n_j-q}}\gamma'^j_{\sigma^{n_j-q}}\cdots\gamma'^{k-m-1}_{\sigma^{m-i}}}\bigg)^{(-1)^{|\vec{n}|+j+q}}$$

and

$$\eta^l_{\alpha_0\cdots\alpha_{k-m-l-1}} = \sum_{i=0}^{m} \sum_{\vec{\sigma}\in F(m)} \sum_{\vec{n}\in P_{k-m-l-1}(i)} \sum_{j=0}^{k-m-l-1} \sum_{q=0}^{n_{j+1}-n_j} (-1)^{|\vec{n}|+j+q} \times \int_{\sigma^{m-i}} \omega^{m+l-i}_{\gamma^0_{\sigma^m}\cdots\gamma^j_{\sigma^{n_j-q}}\gamma'^j_{\sigma^{n_j-q}}\cdots\gamma'^{k-m-l-1}_{\sigma^{m-i}}}.$$

Here, an index $\gamma'^r_{\sigma^s} = (\alpha_r, \phi'(\sigma^s)) \in I \times J$.

By a method similar to that in the proof of Theorem 3.4, we have the following lemma.

Lemma 3.3. The map H is a homotopy operator between $\psi_{K,\phi}$ and $\psi_{K,\phi'}$; that is,

$$\psi_{K,\phi'} - \psi_{K,\phi} = H \circ D + (-1)^m D \circ H. \qquad \square$$

Now we recall that the morphism of complexes

$$\begin{array}{ccccccc}
\underline{\mathbb{C}}^*_X & \xrightarrow{d\log} & \underline{A}^1_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & \underline{A}^p_X \\
 & & & & & & \big\downarrow{\scriptstyle d} \\
 & & & & & & \underline{A}^{p+1}_X
\end{array}$$

induces the homomorphism

$$d : H^p\left(X, \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \underline{A}^1_X \longrightarrow \cdots \longrightarrow \underline{A}^p_X\right) \longrightarrow A^{p+1}(X),$$

where $A^{p+1}(X)$ is the space of complex-valued $(p+1)$-forms.

Theorem 3.4. (i) The integration map $\psi^\sharp = \psi^\sharp_{K,\phi}$ is independent of all choices used in the construction.

(ii) For $k = p$ the following diagram is commutative:

$$\begin{array}{ccc}
H^p\left(X \times M, \underline{\mathbb{C}}^*_{X\times M} \xrightarrow{d\log} \underline{A}^1_{X\times M} \longrightarrow \cdots \longrightarrow \underline{A}^p_{X\times M}\right) & \xrightarrow{d} & A^{p+1}(X \times M) \\
\big\downarrow{\scriptstyle \psi^\sharp} & & \big\downarrow{\scriptstyle (-1)^m \pi_*} \\
H^{p-m}\left(X, \underline{\mathbb{C}}^*_X \xrightarrow{d\log} \longrightarrow \underline{A}^1_X \cdots \longrightarrow \underline{A}^{p-m}_X\right) & \xrightarrow{d} & A^{p-m+1}(X)
\end{array}$$

where π_* is ordinary integration along the fiber for the product bundle $\pi : X \times M \to X$. $\square$

Proof. (i) We need to prove that $\psi^\sharp_{K,\phi}$ is independent of the choice of good covering and cocycle, which represents the same cohomology class, triangulation of the fiber M, and index map. At first, Lemma 3.3 implies that $\psi^\sharp_{K,\phi}$ is independent of the choice of index map. Moreover, considering a common subdivision, it shows that $\psi^\sharp_{K,\phi}$ is independent of the choice of triangulation of M.

By standard argument, it follows from Corollary 3.2 that $\psi^\sharp_{K,\phi}$ is independent of the choice of covering and cocycle.

(ii) We recall that for a Čech cocycle $(f, \omega^1, \dots, \omega^p) \in C^p(\mathfrak{W}, \underline{\mathbb{C}}^*_{X\times M} \xrightarrow{d\log} \underline{A}^1_{X\times M} \to \cdots \to \underline{A}^p_{X\times M})$, the morphism d is defined by

$$d\big(f, \omega^1, \dots, \omega^p\big) = d\omega^p,$$

where $d\omega^p$ denotes the global $(p+1)$-form, which is given by piecing together $\{d\omega^p_{(\alpha,\beta)}\}_{(\alpha,\beta)\in I\times J}$. Hence, to prove (ii), it is sufficient that for any cover $V_{\alpha_0 l} \in \mathfrak{V}$, we show

$$d\theta^{p-m}_{\alpha_0} = (-1)^m \sum_{\sigma^m} \int_{\sigma^m} d\omega^p_{\gamma^0_{\sigma^m}},$$

where we put $(g, \theta^1, \dots, \theta^{p-m}) = \psi_{K,\phi}(f, \omega^1, \dots, \omega^p)$.

Now, by the definition of the integration map $\psi_{K,\phi}$, we have

$$\begin{aligned} d\theta^{p-m}_{\alpha_0} &= \sum_{i=0}^{m} \sum_{\vec{\sigma}\in\mathcal{F}(i)} d\int_{\sigma^{m-i}} \omega^{p-i}_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i}}} \\ &= \sum_{i=0}^{m} \sum_{\vec{\sigma}\in\mathcal{F}(i)} (-1)^{m-i}\bigg(\int_{\sigma^{m-i}} d\omega^{p-i}_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i}}} \\ &\qquad\qquad - \int_{\partial\sigma^{m-i}} \omega^{p-i}_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i}}} \bigg) \\ &= \sum_{\sigma^m} (-1)^m \int_{\sigma^m} d\omega^p_{\gamma^0_{\sigma^m}} + \sum_{i=1}^{m} \sum_{\vec{\sigma}\in\mathcal{F}(i)} (-1)^{m-i} \int_{\sigma^{m-i}} d\omega^{p-i}_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i}}} \\ &\quad - \sum_{i=0}^{m-1} \sum_{\vec{\sigma}\in\mathcal{F}(i)} (-1)^{m-i} \int_{\partial\sigma^{m-i}} \omega^{p-i}_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i}}}. \end{aligned}$$

Next, we show that the last two terms on the right-hand side cancel. On the one hand, by the cocycle condition, we have

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{\vec{\sigma}\in\mathcal{F}(i)} (-1)^{m-i} \int_{\sigma^{m-i}} d\omega^{p-i}_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i}}} \\ &\quad = \sum_{i=1}^{m} \sum_{\vec{\sigma}\in\mathcal{F}(i)} (-1)^{m+1} \int_{\sigma^{m-i}} \big(\delta\omega^{p-i+1}\big)_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i}}} \\ &\quad = \sum_{i=1}^{m} \sum_{\vec{\sigma}\in\mathcal{F}(i)} (-1)^{m-i+1} \int_{\sigma^{m-i}} \omega^{p-i+1}_{\gamma^0_{\sigma^m}\gamma^0_{\sigma^{m-1}}\cdots\gamma^0_{\sigma^{m-i+1}}\widehat{\gamma^0_{\sigma^{m-i}}}}. \end{aligned}$$

In the last line we use the fact that if $s > m - i$, then for each partial flag

$$\sigma^{m-i} \subset \cdots \subset \sigma^{s-1} \subset \widehat{\sigma^s} \subset \sigma^{s+1} \subset \cdots \subset \sigma^m$$

there exist just two flags

$$\sigma^{m-i} \subset \cdots \subset \sigma^{s-1} \subset \sigma^s_+ \subset \sigma^{s+1} \subset \cdots \subset \sigma^m$$

and

$$\sigma^{m-i} \subset \cdots \subset \sigma^{s-1} \subset \sigma^s_- \subset \sigma^{s+1} \subset \cdots \subset \sigma^m$$

such that the terms corresponding to the above two flags cancel in pairs.

On the other hand, we have

$$\begin{aligned}
&\sum_{i=0}^{m-1} \sum_{\vec{\sigma} \in F(i)} (-1)^{m-i} \int_{\partial \sigma^{m-i}} \omega^{p-i}_{\gamma^0_{\sigma^m} \gamma^0_{\sigma^{m-1}} \cdots \gamma^0_{\sigma^{m-i}}} \\
&\quad = \sum_{i=0}^{m-1} \sum_{\vec{\sigma} \in F(i+1)} (-1)^{m-i} \int_{\sigma^{m-i-1}} \omega^{p-i}_{\gamma^0_{\sigma^m} \gamma^0_{\sigma^{m-1}} \cdots \gamma^0_{\sigma^{m-i}}} \\
&\quad = \sum_{i'=1}^{m} \sum_{\vec{\sigma} \in F(i')} (-1)^{m-i'+1} \int_{\sigma^{m-i'}} \omega^{p-i'+1}_{\gamma^0_{\sigma^m} \gamma^0_{\sigma^{m-1}} \cdots \gamma^0_{\sigma^{m-i'+1}}},
\end{aligned}$$

where we put $i' = i + 1$.

Therefore, we find

$$d\theta^{p-m}_{\alpha_0} = (-1)^m \sum_{\sigma^m} \int_{\sigma^m} d\omega^p_{\gamma^0_{\sigma^m}},$$

which completes the proof of Theorem 3.4(ii). ■

Finally, we note that all arguments hold for any oriented fiber bundle $\pi : E \to X$ with a fiber M in place of the product bundle $\pi : X \times M \to X$; indeed, we only need to use an open covering $\{\varphi_\alpha^{-1}(U_\alpha \times V_\beta)\}_{(\alpha,\beta) \in I \times J}$ of the total space E in place of $\mathfrak{W} = \{U_\alpha \times V_\beta\}_{(\alpha,\beta) \in I \times J}$, where $\{V_\beta\}_{\beta \in J}$ is a good cover of the fiber M, and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is a trivialization of the fiber bundle $\pi : E \to X$ such that $\{U_\alpha\}_{\alpha \in I}$ is a good cover of the base space X.

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Gomi: Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1 Meguro-ku, Tokyo 153-8914, Japan; kgomi@ms.u-tokyo.ac.jp

Terashima: Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1 Meguro-ku, Tokyo 153-8914, Japan; tera@ms.u-tokyo.ac.jp